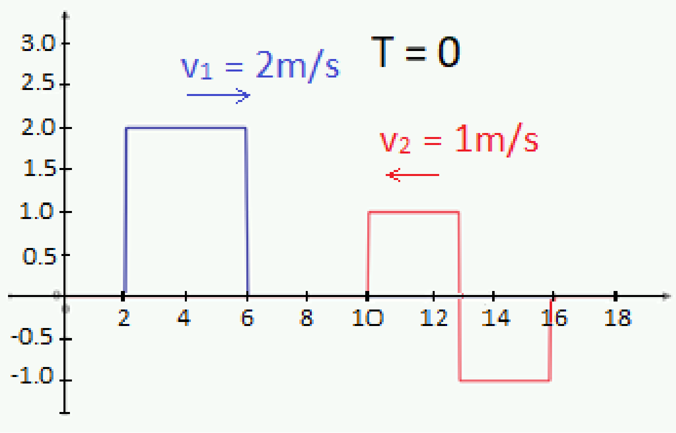
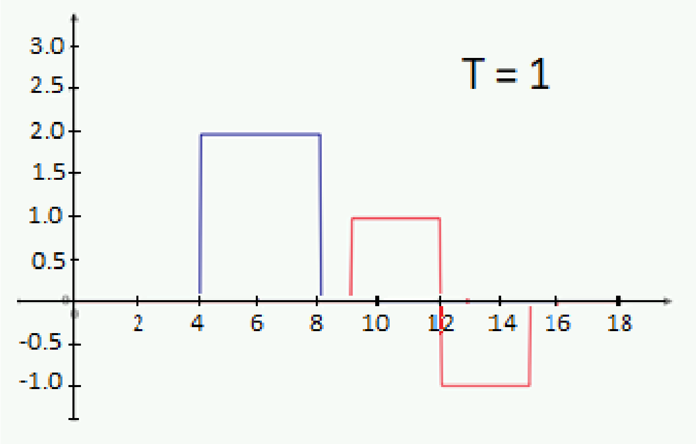
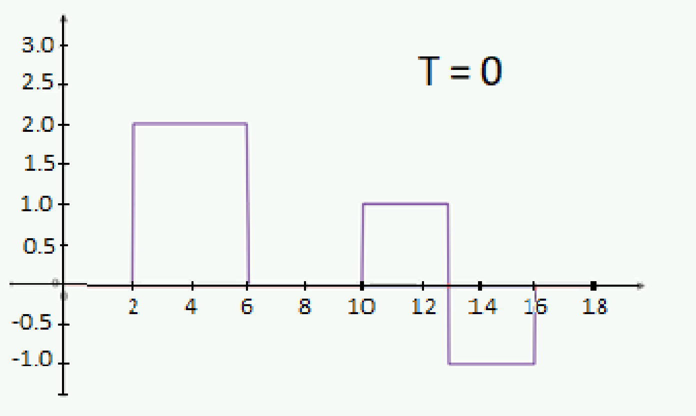
C.1 Wave transmission and reflection

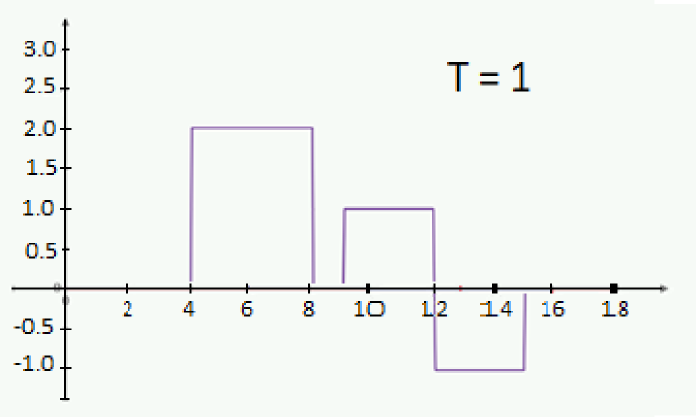
Before we discuss transmission and reflection, we have to discuss what happens when two waves traveling down the same medium run into each other. On the left is plotted two waves approaching each other, and on the right is what the string will look like. We'll suppose the blue wave pulse is traveling right at 2m/s, and the red wave pulse is traveling at 1m/s left. Following website provides a good illustration (under linear superposition). http://www.animations.physics.unsw.edu.au/jw/waves_superposition_reflection.htm



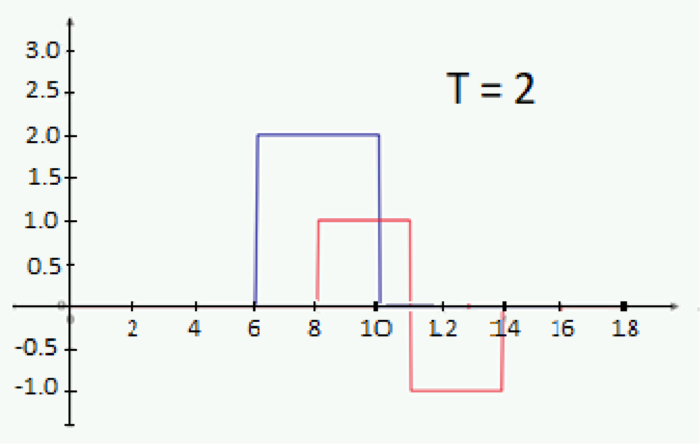


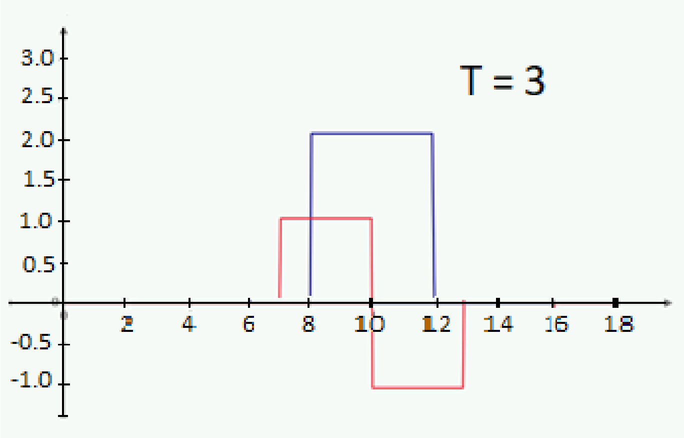
At t = 1s, they'll be closer to each other.

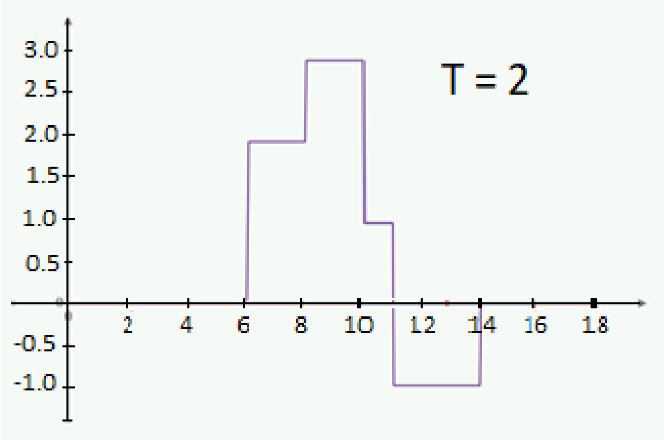




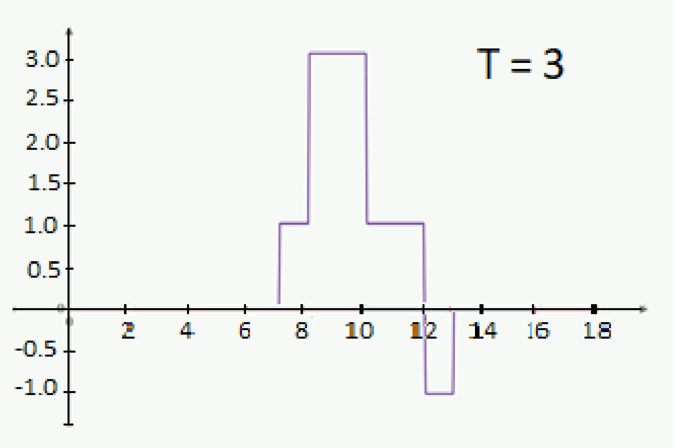
C.1 Wave transmission and reflection



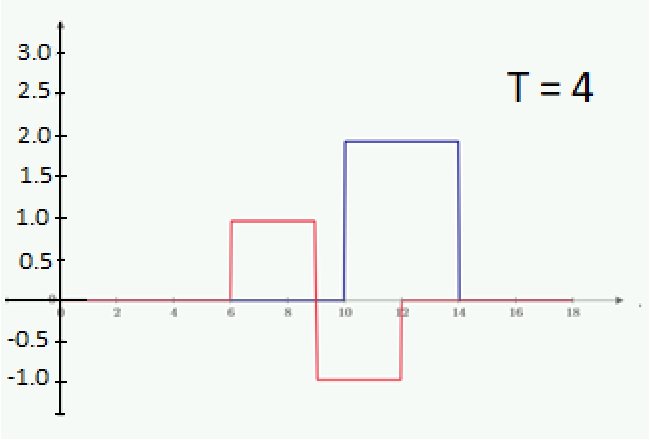


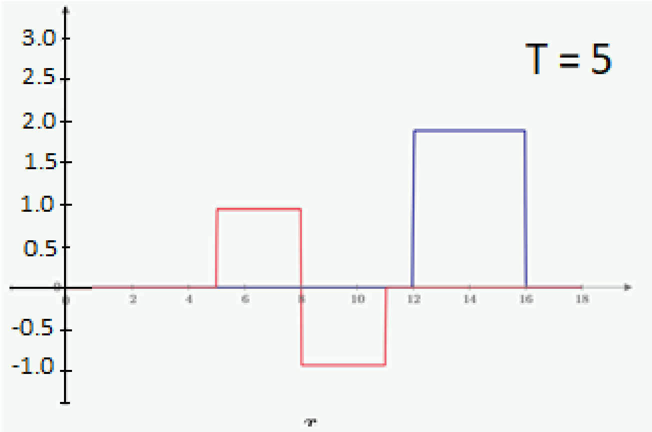


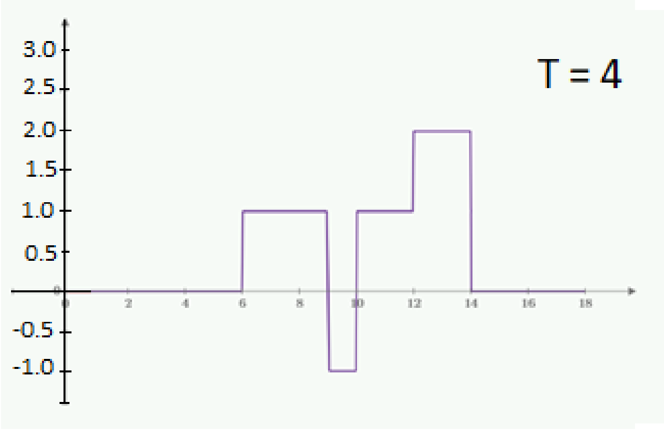
What waves do when they overlap depends on the medium. For linear mediums (one's that obey our wave equation developed at the beginning of part B), the waves will *superimpose* on each other, i.e., the shape the string will take is the *sum* of the two waves' displacements.

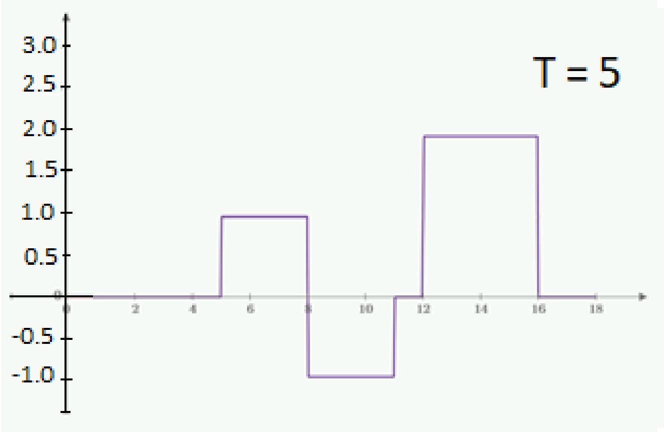


C.1 Wave transmission and reflection





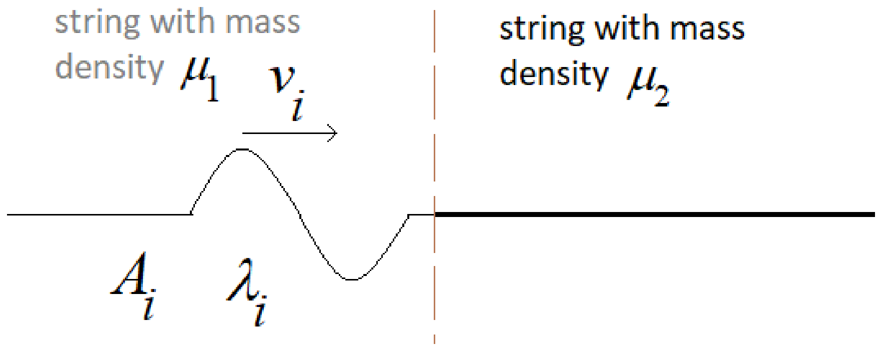




Once the waves pass each other, they no longer overlap, and the string's shape will just remain the shape of the individual waves.

C. Wave transmission and reflection (1D)

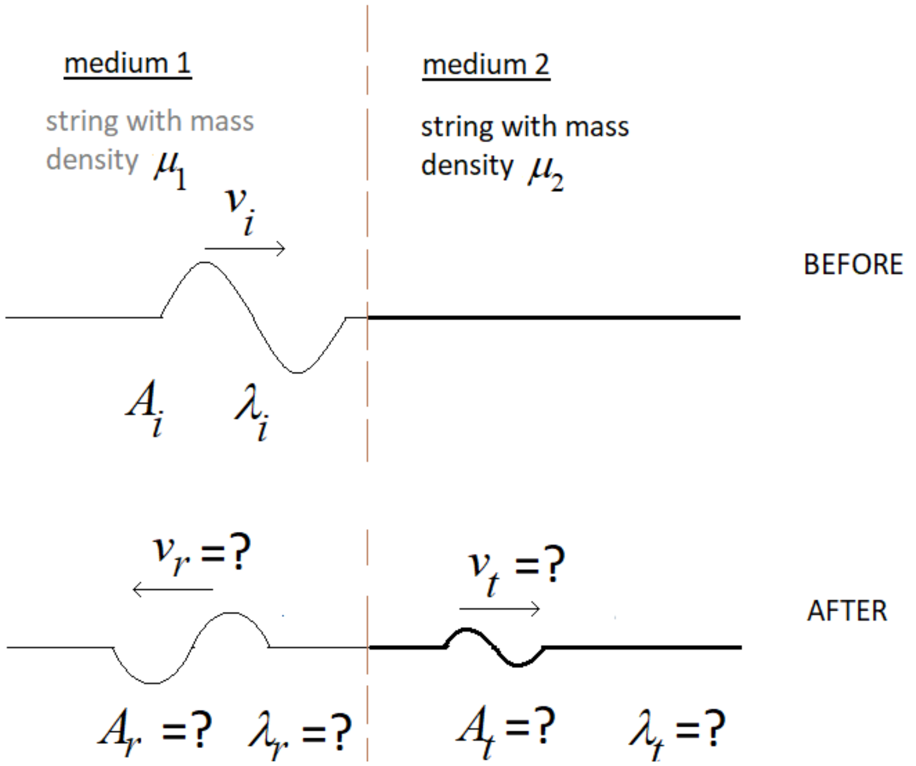
Now we'd like to consider what happens when a wave traveling in one medium enters into another medium, e.g. when light waves in air penetrate into a glass window, and then emerge into air again, or when a sound wave in air penetrate a wall and emerge into air on the other side (how hear people on the other side of a wall). First we'll consider the situation in 1D, since it's easier to mathematically analyze this situation. And the easiest 1D situation to analyze in detail is the case of a string. Suppose we tie two ropes together with different mass densities, μ_1, μ_2 and then we send a pulse down string 1. We'd like to know what will happen as the wave enters the second string.



A good visualization is provided by the following website.

http://physics.usask.ca/~hirose/ep225/animation/reflection/anim-reflection.htm

C.1 Wave transmission and reflection (1D)



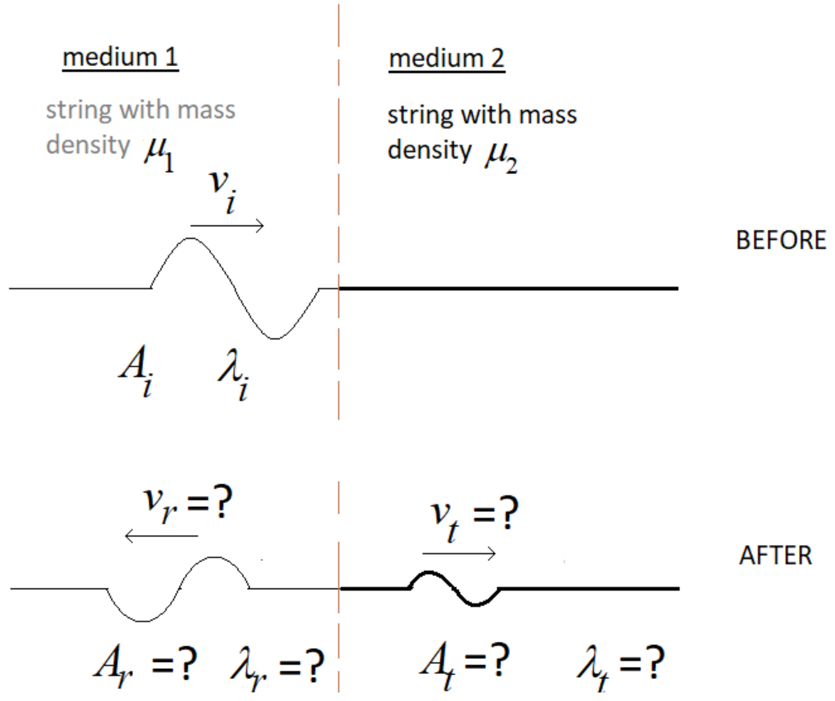
So we saw that when a wave hits the interface, the wave's energy will split into a transmitted wave and a reflected wave. We'd like to analyze these waves' properties. Specifically, we want to know how the reflected and transmitted waves' f's, v's, λ's, and A's, will compare to those of the incident wave.

Frequency

The incident wave is the direct cause of the reflected and transmitted waves. And so the rate which the incident waves hit the interface, is equal to the rate at which the reflected and transmitted waves will be generated. So:

$$f_i = f_r = f_t \longrightarrow f_1 = f_2$$

C.1 Wave transmission and reflection (1D)



We're defining the 'index of refraction', **n**, which characterizes how difficult it is for a wave to penetrate into the medium.

$n = \sqrt{\mu}$

velocity

We have a formula for the velocity of each wave already. We know:

$$v = \sqrt{\frac{T}{\mu}}$$

Since the reflected wave is traveling in the same rope as the incident wave, it will have the same velocity. But the transmitted wave velocity depends....first, we can say that the tension in the two ropes is the same (if it were otherwise, the interface would experience a net force, and accelerate one way or the other). Then to compare we can write:

$$\frac{v_t}{v_i} = \frac{\sqrt{T/\mu_2}}{\sqrt{T/\mu_i}} = \frac{\sqrt{\mu_1}}{\sqrt{\mu_2}}$$

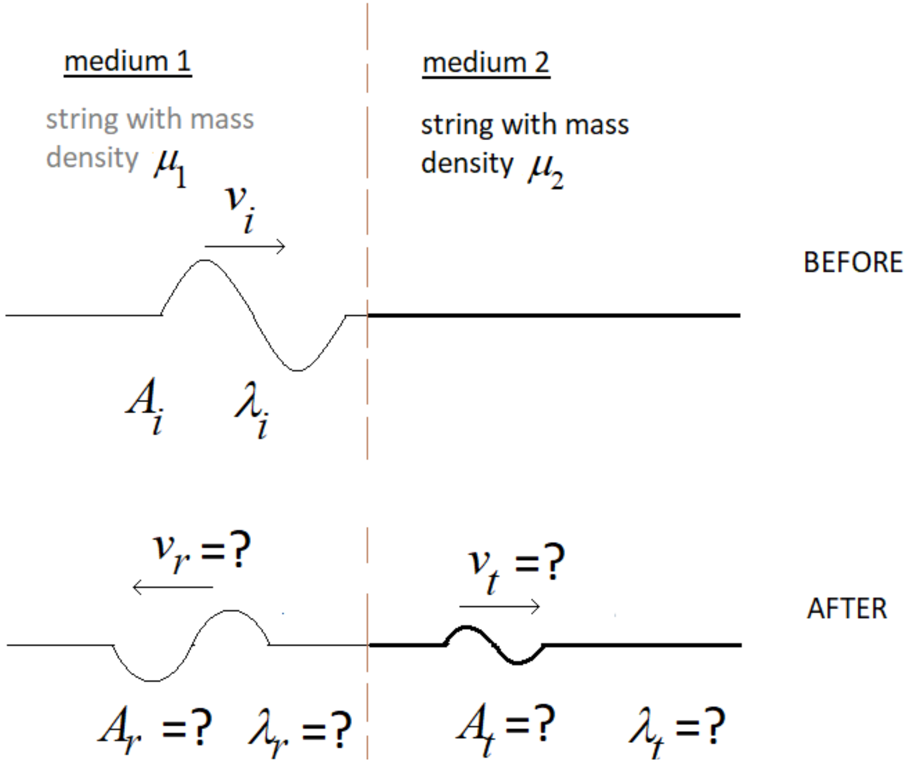
$$\sqrt{\mu_1} v_i = \sqrt{\mu_2} v_t$$

$$n_1 v_i = n_2 v_t$$

$$n_1 v_1 = n_2 v_2$$

Equation implies that if n goes up, then v goes down.

C.1 Wave transmission and reflection (1D)



wavelength

Since we know how frequency and speed vary, we know how wavelength varies. First, since f and v are the same for the incident and reflected waves, their wavelengths are the same too. Then to relate the incident and transmitted waves….

$$n_1 v_i = n_2 v_t$$

$$\frac{n_1 v_i}{f_i} = \frac{n_2 v_t}{f_i}$$

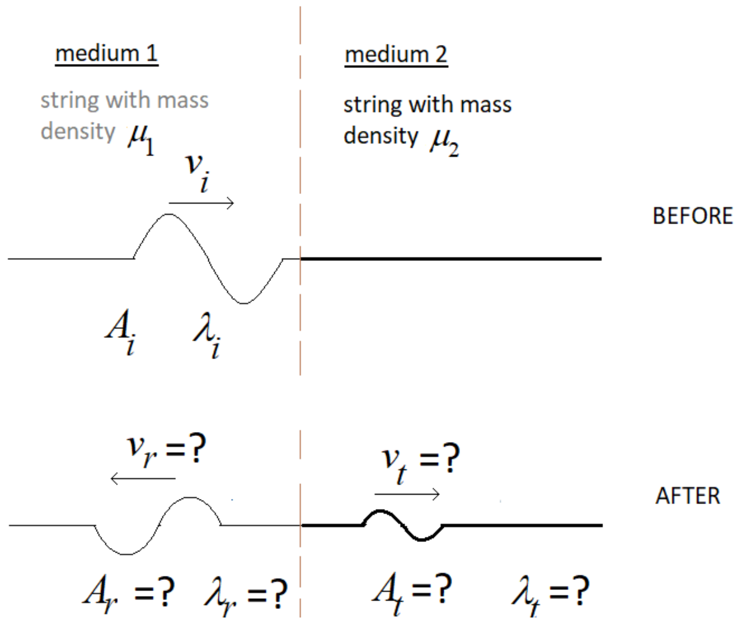
$$\frac{n_1 v_i}{f_i} = \frac{n_2 v_t}{f_t}$$

$$n_1 \lambda_i = n_2 \lambda_t$$

$$n_1 \lambda_1 = n_2 \lambda_2$$

Equation implies that if n goes up, λ will go down. Basically, higher n will slow down v and make the wave bunch up, like cars do when they hit traffic.

C.1 Wave transmission and reflection (1D)



Amplitude

Now the hard part. Working out how the amplitudes relate requires a detailed analysis of how the waves are attempting to displace the interface. So we will write a formula for the displacement of the string. Note we use fact that f's (and so ω's) are same for all waves, λ's (and so k's) are same for incident and reflected waves. And v's are same for incident and reflected waves.

$$
\begin{aligned}
D_1(x,t) &= D_{incident}(x,t) + D_{reflection}(x,t) \\
&= A_i \sin\left[k_i(x - v_i t)\right] + A_r \sin\left[k_r(x + v_r t)\right] \\
&= A_i \sin\left[k_1(x - v_1 t)\right] + A_r \sin\left[k_1(x + v_1 t)\right] \\
&= A_i \sin\left[k_1 x - \omega_1 t\right] + A_r \sin\left[k_1 x + \omega_1 t\right]
\end{aligned}
$$

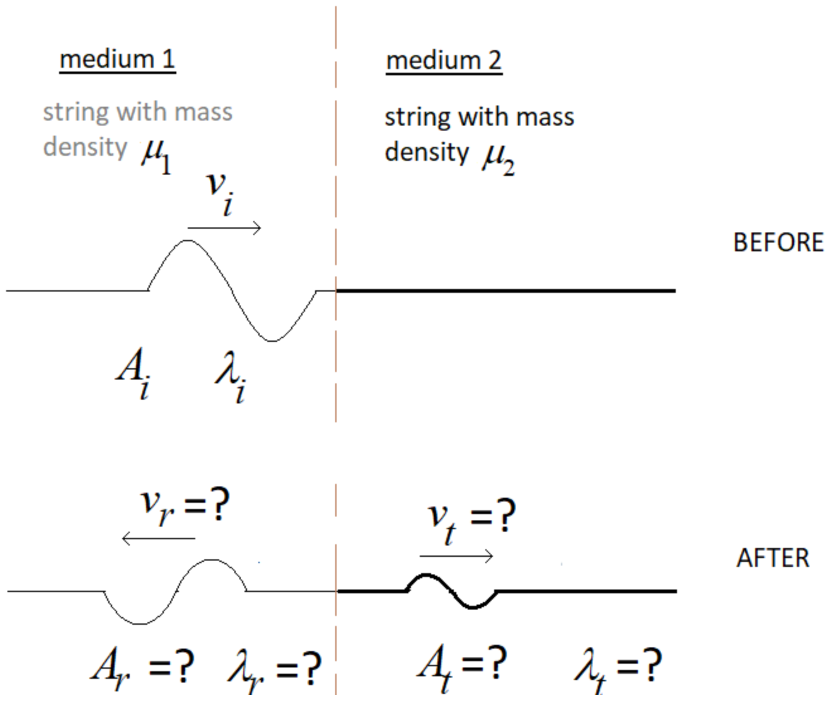
$$
\begin{aligned}
D_2(x,t) &= D_{transmission}(x,t) \\
&= A_t \sin\left[k_t(x - v_t t)\right] \\
&= A_t \sin\left[k_2(x - v_2 t)\right] \\
&= A_t \sin\left[k_2 x - \omega_2 t\right] \\
&= A_t \sin\left[k_2 x - \omega_1 t\right]
\end{aligned}
$$

Then we impose the boundary conditions, namely that the displacement of the string must be the same on each side of the interface (string can't break), and that the slope of the string must be the same as well (no kinks). We'll take the interface to be at x = 0.

$$
\begin{aligned}
&D_1(x=0,t) = D_2(x=0,t) \\
&A_i \sin\left[k_1(0) - \omega_1 t\right] + A_r \sin\left[k_1(0) + \omega_1 t\right] = A_t \sin\left[k_2(0) - \omega_1 t\right] \\
&A_i \sin\left[-\omega_1 t\right] + A_r \sin\left[\omega_1 t\right] = A_t \sin\left[-\omega_1 t\right] \\
&-A_i \sin\left[\omega_1 t\right] + A_r \sin\left[\omega_1 t\right] = -A_t \sin\left[\omega_1 t\right] \\
&-A_i + A_r = -A_t
\end{aligned}
$$

$$
\begin{aligned}
&\frac{d}{dx} D_1(x=0,t) = \frac{d}{dx} D_2(x=0,t) \\
&A_i k_1 \cos\left[k_i(0) - \omega_1 t\right] + A_r k_1 \cos\left[k_1(0) + \omega_1 t\right] = A_t k_2 \cos\left[k_2(0) - \omega_1 t\right] \\
&A_i k_1 \cos\left[\omega_1 t\right] + A_r k_1 \cos\left[\omega_1 t\right] = A_t k_2 \cos\left[\omega_1 t\right] \\
&A_i k_1 + A_r k_1 = A_t k_2
\end{aligned}
$$

C.1 Wave transmission and reflection (1D)



Amplitude

If we solve these two equations for A_r and A_t we will get:

$$A_r = \frac{1 - k_2/k_1}{1 + k_2/k_1} A_i = \frac{1 - \lambda_1/\lambda_2}{1 + \lambda_1/\lambda_2} A_i = \frac{1 - n_2/n_1}{1 + n_2/n_1} A_i = \frac{n_1 - n_2}{n_1 + n_2} A_i$$

$$A_t = \frac{2}{1 + k_2/k_1} A_i = \frac{2}{1 + \lambda_1/\lambda_2} A_i = \frac{2}{1 + n_2/n_1} A_i = \frac{2n_1}{n_1 + n_2} A_i$$

So there we go:

$$A_r = \frac{n_1 - n_2}{n_1 + n_2} A_i$$

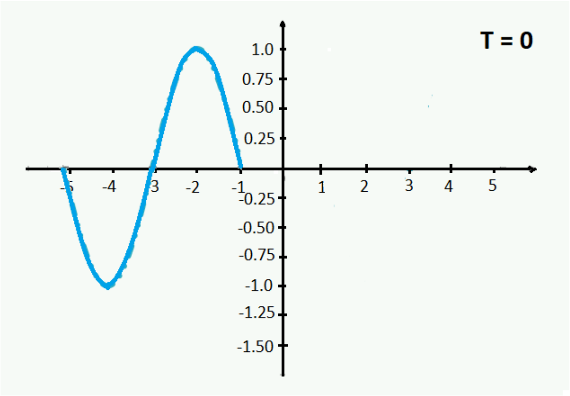
$$A_t = \frac{2n_1}{n_1 + n_2} A_i$$

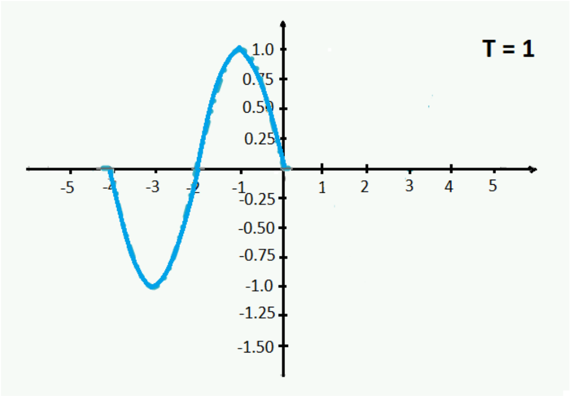
Note the following, if you please ….

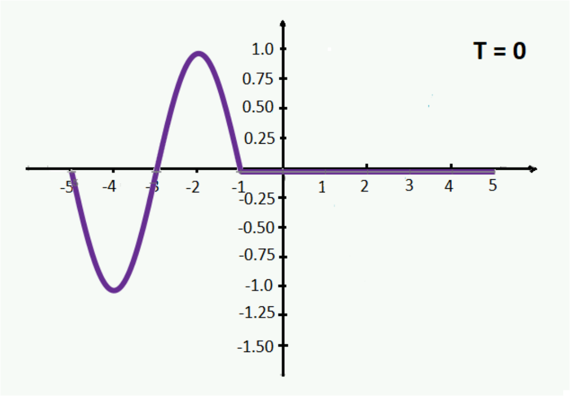
- $n_1 < n_2 \rightarrow A_r$ is negative (wave upside down)
- $n_1 = n_2 \rightarrow A_r = 0$ (no reflection)
- $n_1 > n_2 \rightarrow A_r$ is positive (wave right side up)

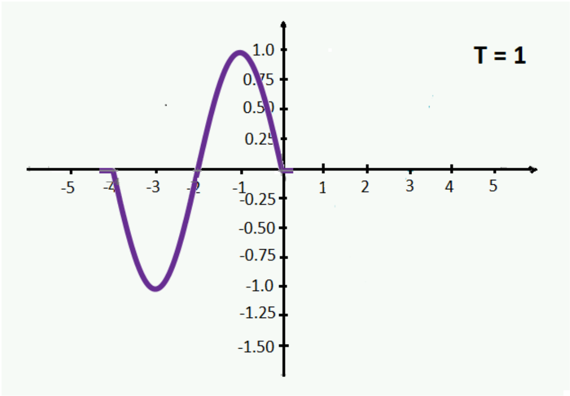
C.1 Wave transmission and reflection (1D)

I've got a great idea! We're going to graph what happens as a wave enters into a new medium. Let's say we have our incident sinusoidal pulse traveling to the right down medium 1 ($x < 0$, $n_1 = 1$) at speed $v_1 = 1$m/s. And then we'll see what will happen as it hits medium 2 ($x > 0$, $n_2 = 2$). In the top row we have the incident wave, and in the bottom row, we have what the string will look like.

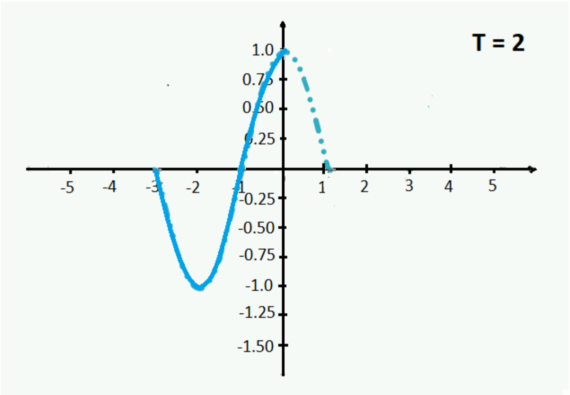




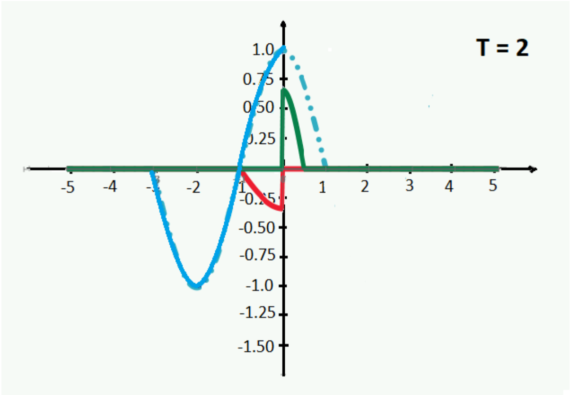




C.1 Wave transmission and reflection (1D)



If medium 2 were the same as medium 1, then this is what the string would look like. The ‘faux’ transmission (the part of the wave that extends beyond x = 0) is a useful aid for constructing the reflected and actual transmitted waves.



So transmitted wave (green) should be 2/3 the size of the ‘faux’ transmission, and have traveled only 0.5m.

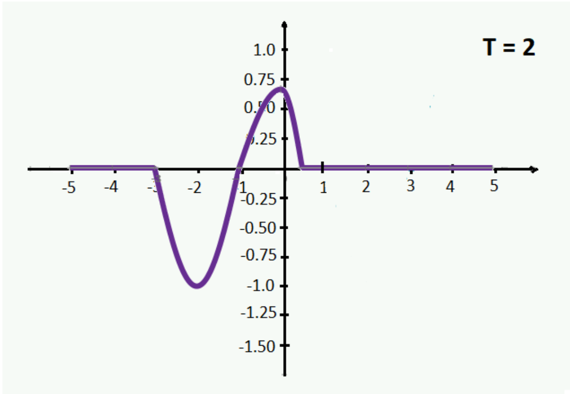
$$A_t = \frac{2n_1}{n_1 + n_2} A_i = \frac{2(1)}{1+2} A_i = \frac{2}{3} A_i$$

$$n_1 v_i = n_2 v_t \rightarrow (1)(1) = (2) v_t \rightarrow v_t = 0.5\,\text{m/s}$$

Reflected wave (red) should be -1/3 size of ‘faux’ transmission, and have traveled 1m.

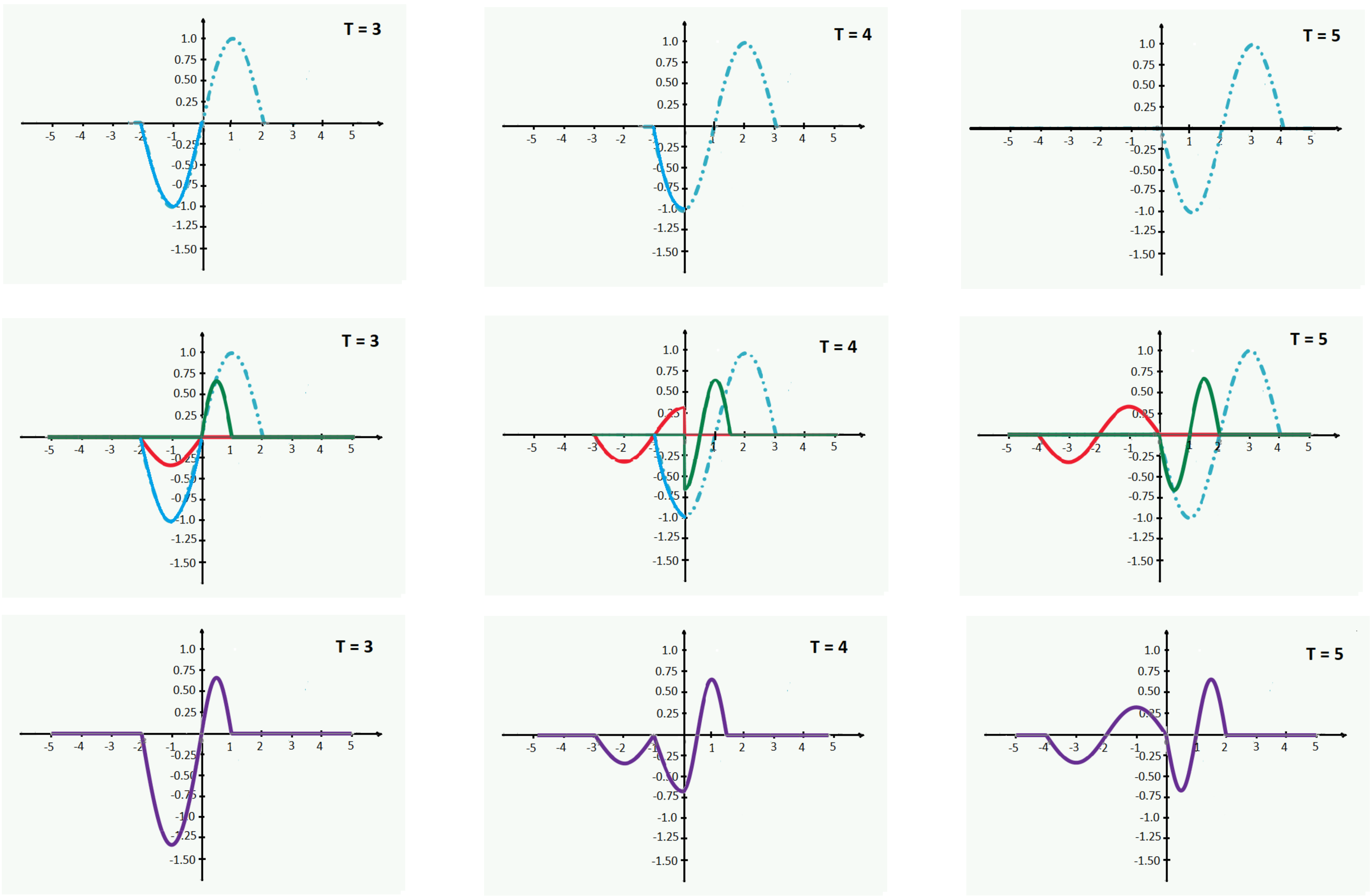
$$A_r = \frac{n_1 - n_2}{n_1 + n_2} A_i = \frac{1-2}{1+2} A_i = -\frac{1}{3} A_i$$

$$v_r = v_i \rightarrow v_r = 1\,\text{m/s}$$

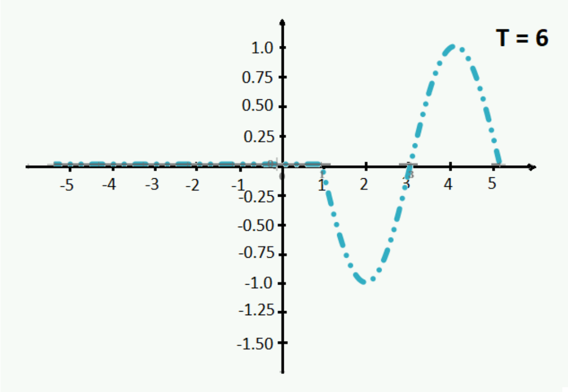


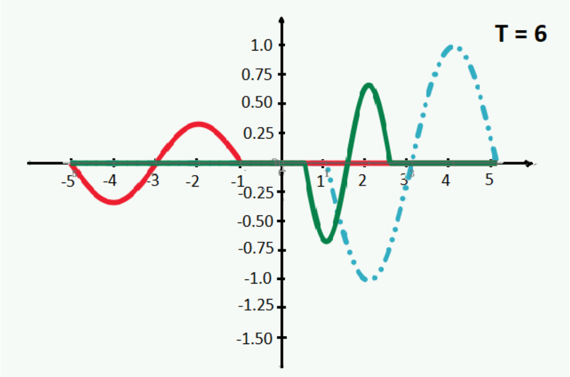
And this is the superposition of the incident and reflected waves. You can see that when you add the two, they match up evenly with the transmitted wave, as was required by those boundary conditions we employed.

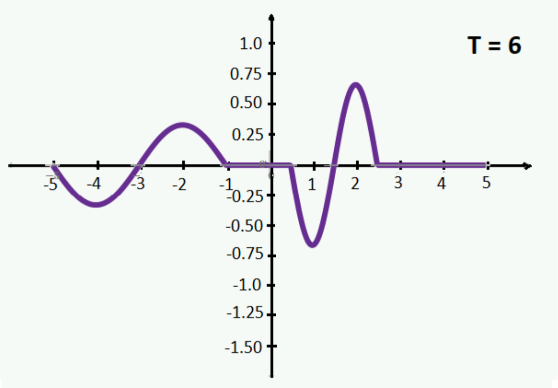
C.1 Wave transmission and reflection (1D)



C.1 Wave transmission and reflection (1D)







The reflected wave will then continue to travel to the left at 1m/s, and the transmitted wave will continue to travel to the right at 0.5m/s. Observe how the wavelength of the transmitted wave is half that of the incident (and reflected) wave, in accordance with the equation: $n_1\lambda_1 = n_2\lambda_2$.

C.1 Wave transmission and reflection (1D)

Now let's consider a more quantitative example. Let's say the moon(light) directly overhead hits your eye (or a pond rather), like a big pizza pie (or like a shiny ball of rock instead). Suppose the light's wavelength is 600nm, its speed is 3×10^8m/s, and its intensity is a mellow 100W/m^2. The index of refraction of light in air is 1, and in water is 1.33. In what follows, assume the relationship n = √μ, is valid for light waves, as it is for string waves (it's not quite, but will actually give us the correct results anyway).

(a) What are the frequencies of the incident, reflected, and transmitted waves?

So the frequency of light in air can be obtained from the velocity and wavelength given:

$$f_i = \frac{v_i}{\lambda_i} = \frac{3\times10^8\,\text{m/s}}{500\times10^{-9}\,\text{m}} = 6\times10^{14}\,\text{Hz}$$

And the frequencies of the reflected and transmitted light rays will be the same.

(b) What are the wavelengths of the incident, reflected, and transmitted waves?

The reflected and incident wavelengths are the same. The transmitted one is given by:

$$n_1\lambda_i = n_2\lambda_t$$

$$(1)(500\,\text{nm}) = (1.33)\lambda_t$$

$$\lambda_t = \frac{500\,\text{nm}}{1.33} = 376\,\text{nm}$$

C.1 Wave transmission and reflection (1D)

(c) What are the velocities of the incident, reflected, and transmitted waves?

The velocities are the same for reflected and incident waves. And for the transmitted we have:

$$n_1 v_i = n_2 v_t$$

$$(1)(3\times10^8 \text{ m/s}) = (1.33)v_t$$

$$v_t = \frac{3\times10^8 \text{ m/s}}{1.33} = 2.25\times10^8 \text{ m/s}$$

Alternatively, we could say $v_t = f_t\lambda_t = (6\times10^{14}\text{ Hz})(376\times10^{-9}\text{ m}) = 2.25\times10^8 \text{ m/s}$

(d) What are the intensities of the incident, reflected, and transmitted waves? Is energy conserved?

The easiest way to get the intensities, when you have one of them, is to form a ratio.

$$\frac{I_r}{I_i} = \frac{\frac{1}{2}\mu_r(A_r\omega_r)^2 v_r}{\frac{1}{2}\mu_i(A_i\omega_i)^2 v_i} = \left(\frac{A_r}{A_i}\right)^2 = \left(\frac{n_1-n_2}{n_1+n_2}\right)^2 = \left(\frac{1-1.33}{1+1.33}\right)^2 = 0.02 \longrightarrow I_r = 0.02 I_i = 0.02(100\text{ W/m}^2) = 2\text{ W/m}^2$$

$$\frac{I_t}{I_i} = \frac{\frac{1}{2}\mu_t(A_t\omega_t)^2 v_t}{\frac{1}{2}\mu_i(A_i\omega_i)^2 v_i} = \frac{\mu_t}{\mu_i}\left(\frac{A_r}{A_i}\right)^2\frac{v_t}{v_i} = \frac{n_2^2}{n_1^2}\left(\frac{2n_1}{n_1+n_2}\right)^2\frac{n_1}{n_2} = \frac{n_2}{n_1}\left(\frac{2n_1}{n_1+n_2}\right)^2 = \left(\frac{1.33}{1}\right)\left(\frac{2(1)}{1+1.33}\right)^2 = 0.98 \longrightarrow I_r = 0.98 I_i = 98\text{ W/m}^2$$

The reflected and transmitted intensities add up to the incident intensity so energy is conserved, as it should be.

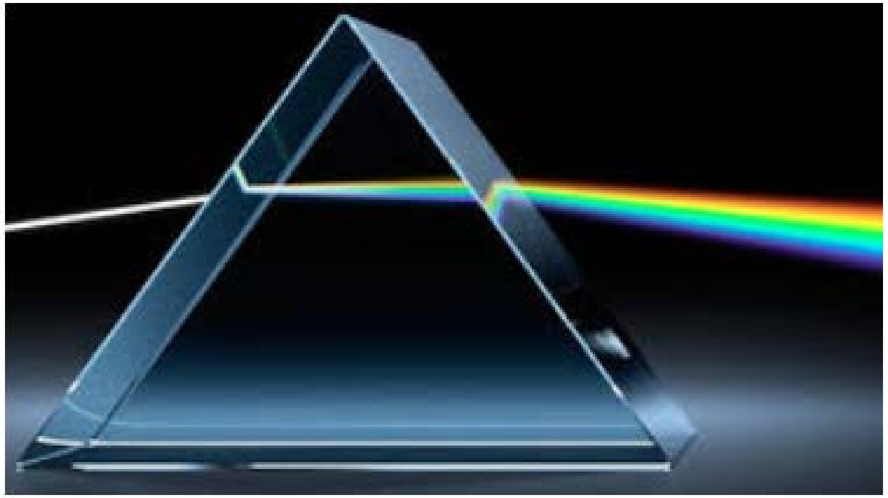
C.2 Wave transmission, reflection (3D)

Things remain preeetty much the same in 3D:

$$\boxed{f_1 = f_2}$$

$$\boxed{n_1\lambda_1 = n_2\lambda_2}$$

$$\boxed{n_1 v_1 = n_2 v_2}$$



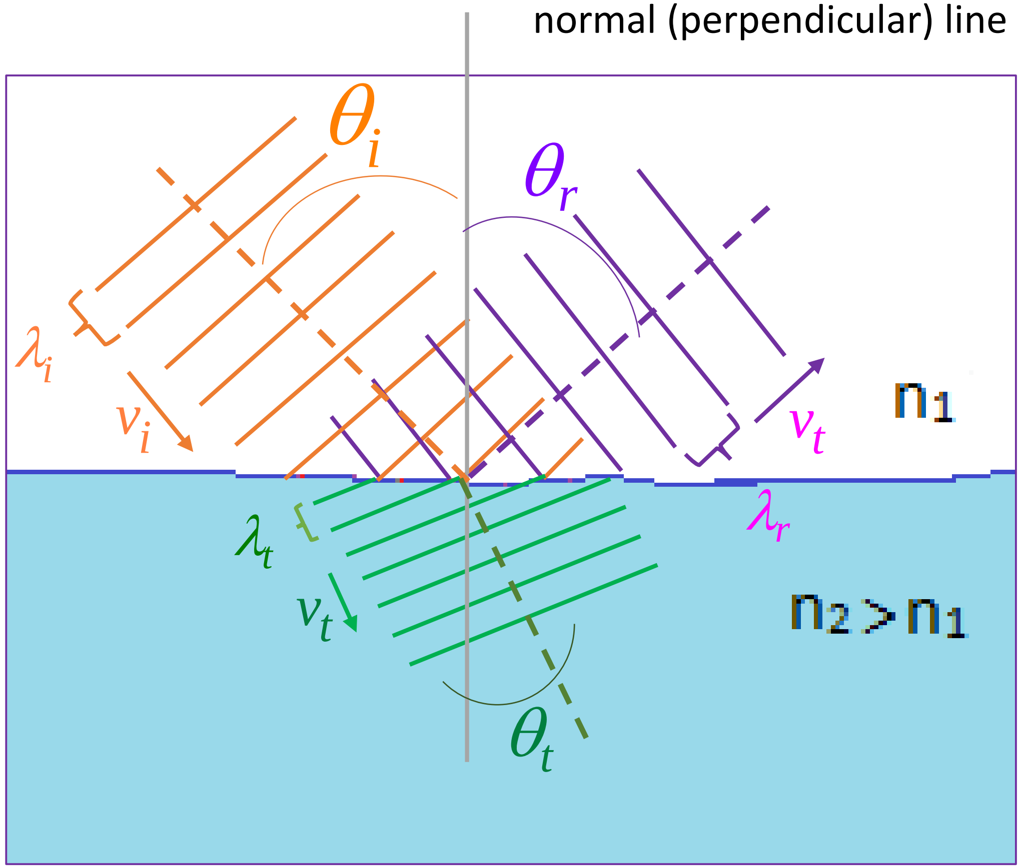
$$\boxed{\begin{aligned} A_r &= \text{complicated function of } (n_1, n_2, \text{angle of incidence})^{\dagger} \cdot A_i \\ A_t &= \text{another complicated function of } (n_1, n_2, \text{angle of incidence}) \cdot A_i \end{aligned}}$$

† It is still the case that if $n_2 > n_1$, the reflected wave will be inverted.

But we can also see, from the picture, that the waves will bend. Let's see if we can make sense of that.

C.2 Wave transmission, reflection (3D)

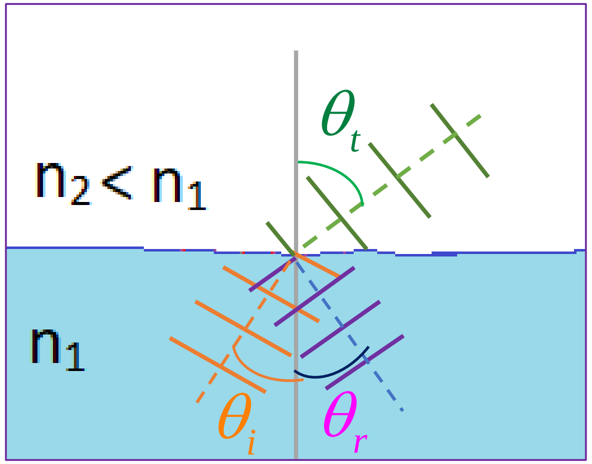
First we'll consider the case of a wave in medium $n_1$entering a medium $n_2 > n_1$.

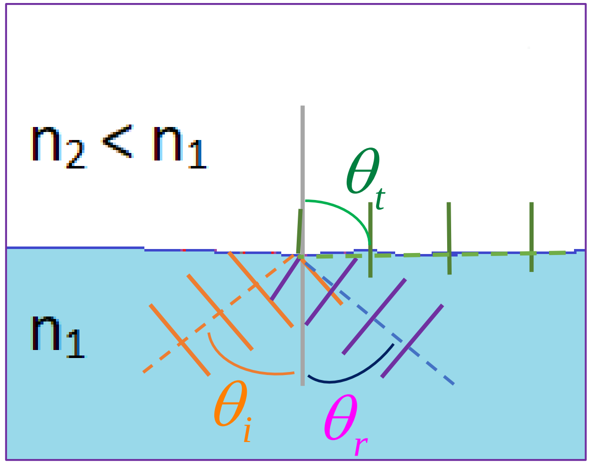


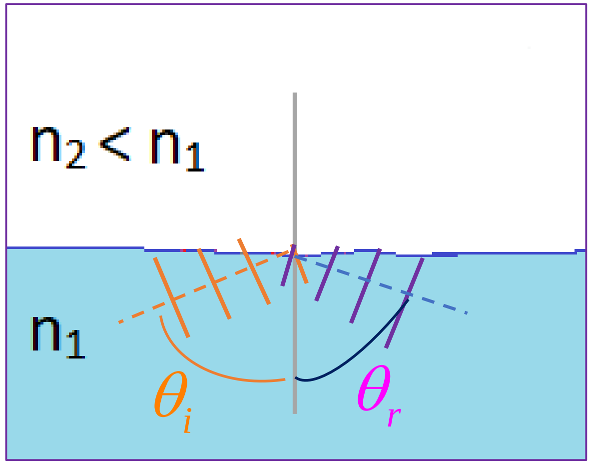
We can see that the reflected wave seems emerges at the same angle as the incident wave. This appears to be a consequence of the equality of the incident and reflected wave speeds. And we'll make note that it must be inverted since $n_2 > n_1$. But the transmitted wave seems to bend *closer* to the normal line. This appears to be a consequence of the dimunition of the transmitted wave's velocity.

C.2 Wave transmission, reflection (3D)

Now let's consider the case of an incident ray coming from n_1 and entering a region $n_2 < n_1$.







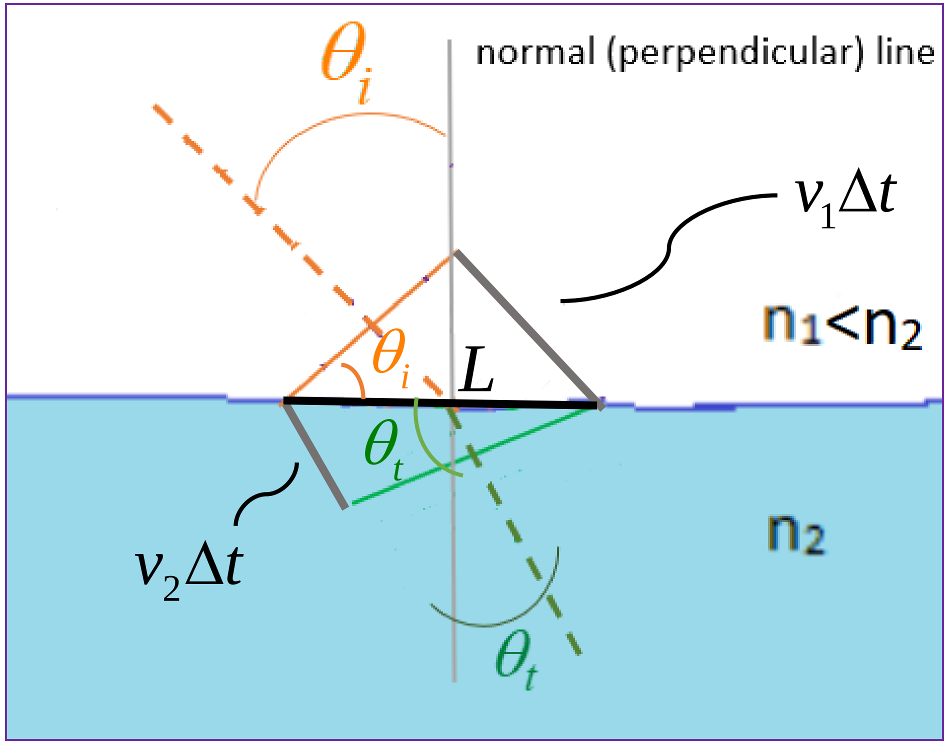
Reflected ray makes same angle as incident again. This time it's not inverted because $n_2 < n_1$. But due to this, transmitted wave velocity is larger than incident, and so it makes a larger angle from the normal line.

If we tilt the incident ray a little more, eventually the transmitted ray will make a 90° angle with the normal. This angle of incidence is called the *critical angle*, Θ_c. What happens if we tilt the incident ray *past* the critical angle?

Then there will be no transmitted ray, and we'll *only* have reflection. This is called ***total internal reflection***, and is what makes fiber optic cables useful, for instance.

C.2 Wave transmission, reflection (3D)

Now let's get more quantitative, and see if we can work out how the transmitted and incident angles relate.



Consider time between last wave front entirely in medium 1 , and first wave front entirely in medium 2.

And then consider the distance traveled by the wave during this time, in grey….

We can relate these to length of the interface, L,

$$L\sin\theta_i = v_1\Delta t \qquad L\sin\theta_t = v_2\Delta t$$

$$\frac{L\sin\theta_i}{L\sin\theta_t} = \frac{v_1\Delta t}{v_2\Delta t}$$

Take the ratio of these equations

$$\frac{\sin\theta_i}{\sin\theta_t} = \frac{n_2}{n_1}$$

because $n_1 v_1 = n_2 v_2$

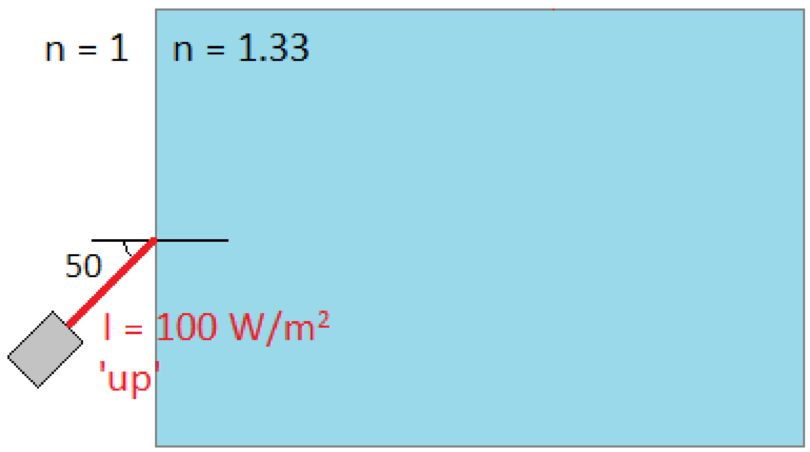
$$\boxed{n_1\sin\theta_i = n_2\sin\theta_t \qquad \text{Snell's law}}$$

Can get critical angle too:

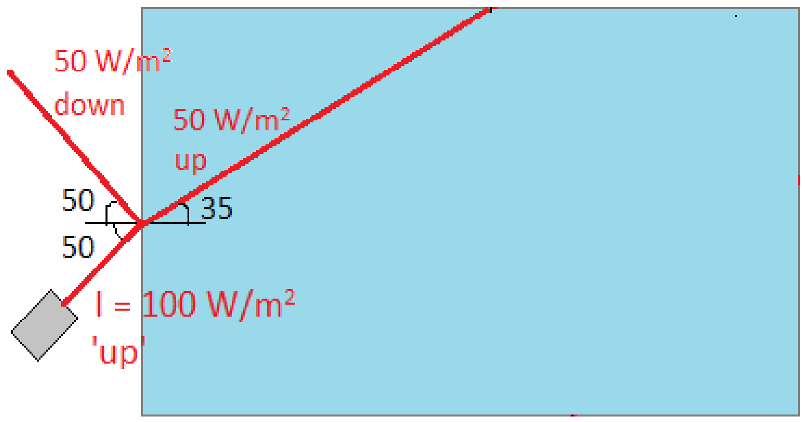
$$n_1\sin\theta_c = n_2\sin 90^\circ$$
$$n_1\sin\theta_c = n_2$$

$$\longrightarrow \boxed{\theta_c = \sin^{-1}\left(\frac{n_2}{n_1}\right)}$$

C.2 Wave transmission, reflection (3D)



Consider the laser beam incident on a water filled transparent box at a 50^0 angle. Say the intensity of the beam is $100 W/m^2$, and is oscillating 'upwards' in the transverse direction. Trace the path of the light ray through the box through four reflections. Give the intensities of the transmitted and reflected rays, assuming they equally share the incident waves energy, and also state the orientation of these waves.



Transmission

$$n_1 \sin\theta_i = n_2 \sin\theta_t$$

$$(1)\sin 50^\circ = (1.33)\sin\theta_t$$

$$\theta_t = \sin^{-1}\left(\frac{\sin 50^\circ}{1.33}\right) = 35^\circ$$

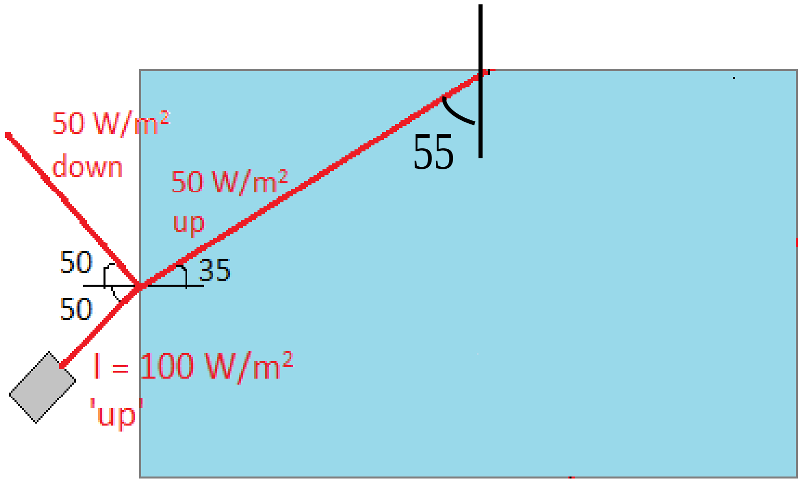
And it will split the intensity, and be 'up'

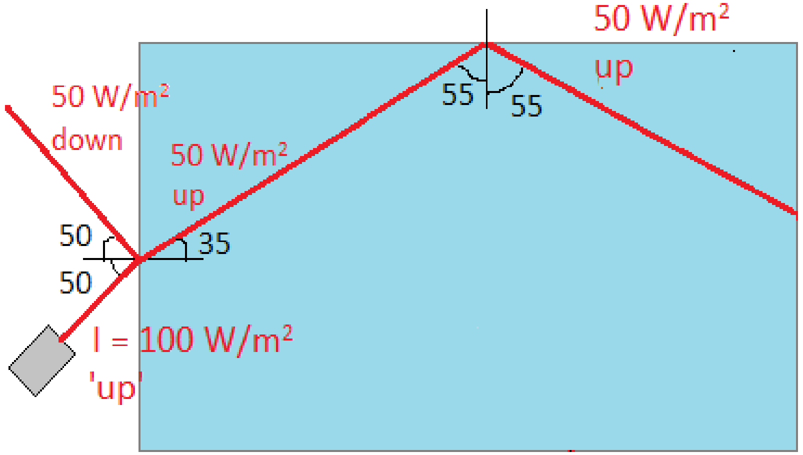
Reflection

$$\theta_r = \theta_i = 50^\circ$$

And it will split the intensity, and be 'down'

C.2 Wave transmission, reflection (3D)





What is the normal line, and angle of incidence?

Transmission

$$n_1 \sin\theta_i = n_2 \sin\theta_t$$

$$(1.33)\sin 55^\circ = (1)\sin\theta_t$$

$$\theta_t = \sin^{-1}\left(1.33\sin 55^\circ\right) = \text{undefined}$$

This indicates we're past the critical angle, which is true since,

$$\theta_c = \sin^{-1}\left(\frac{1}{1.33}\right) = 48^\circ$$

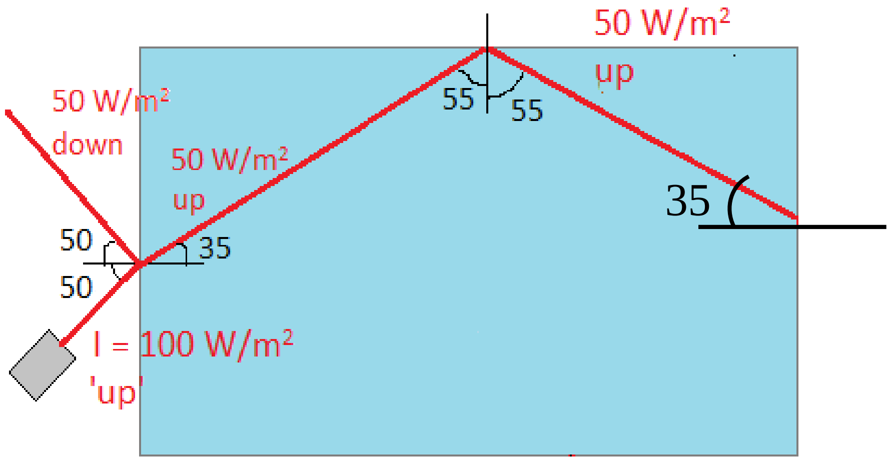
So there is no transmitted ray.

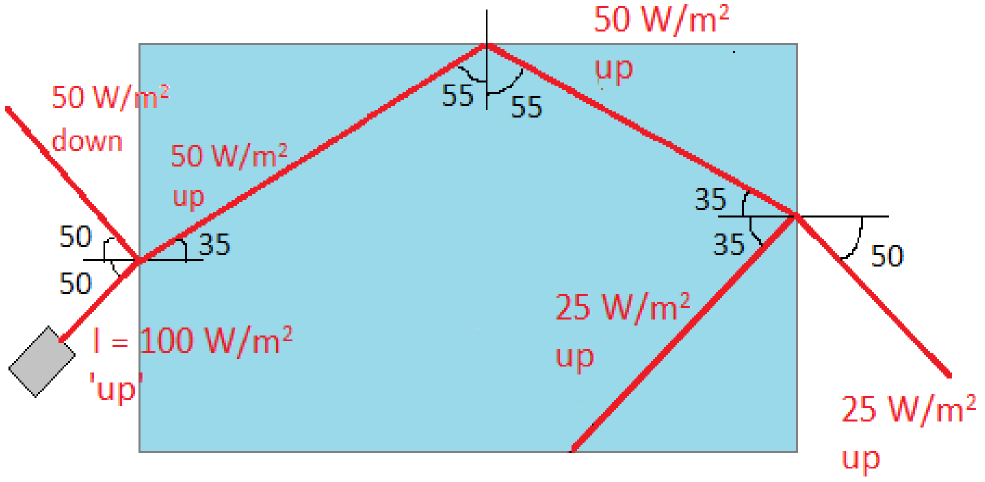
Reflection

$$\theta_r = \theta_i = 55^\circ$$

And it will take all the energy of the incident ray, as well as be 'up', since bouncing off a lower n won't invert it.

C.2 Wave transmission, reflection (3D)





What is the normal line, and angle of incidence?

Transmission

$$n_1 \sin\theta_i = n_2 \sin\theta_t$$

$$(1.33)\sin 35^\circ = (1)\sin\theta_t$$

$$\theta_t = \sin^{-1}\left(1.33\sin 35^\circ\right) = 50°$$

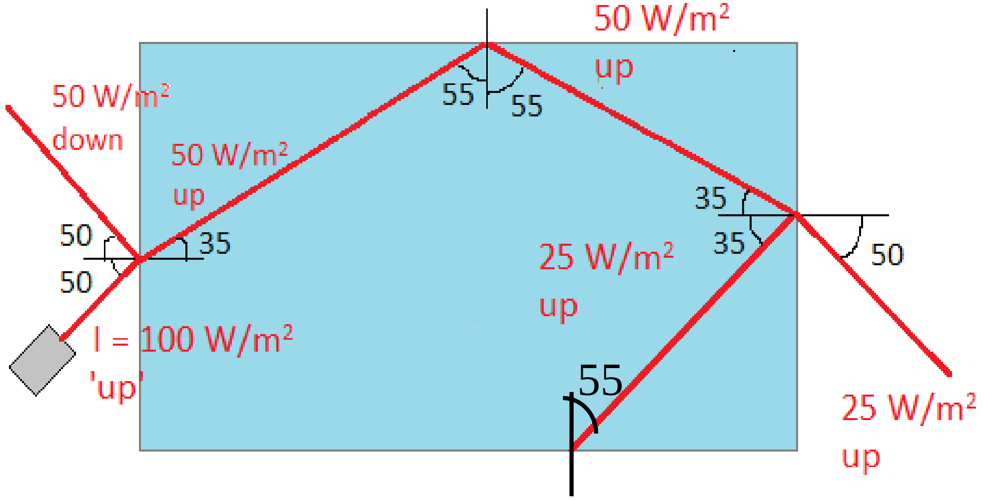
And it will split the intensity and be 'up' – transmitted ray always has same orientation as incident ray.

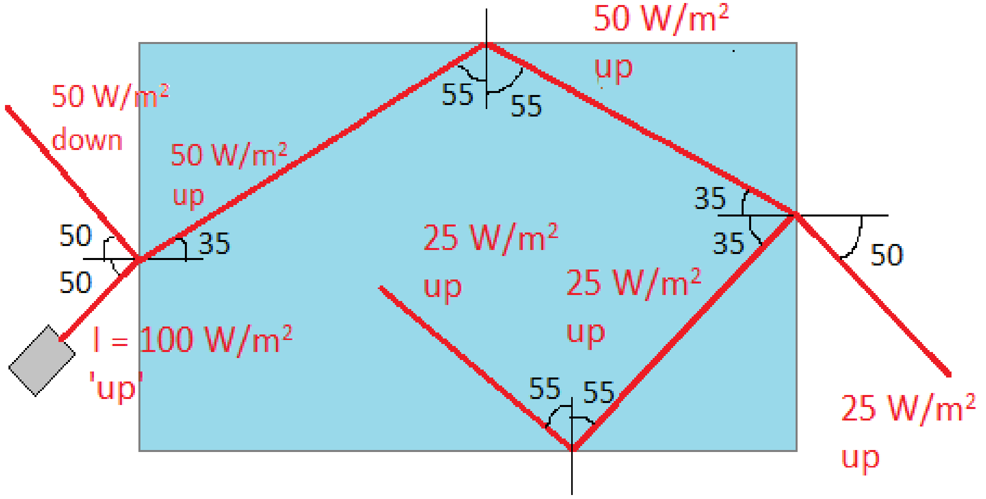
Reflection

$$\theta_r = \theta_i = 35^\circ$$

And it take the remainder of the energy, and be 'up' since it's bouncing off a lower index of refraction.

C.2 Wave transmission, reflection (3D)





What is the normal line, and angle of incidence?

Transmission

$$n_1 \sin\theta_i = n_2 \sin\theta_t$$

$$(1.33)\sin 55^\circ = (1)\sin\theta_t$$

$$\theta_t = \sin^{-1}\left(1.33\sin 55^\circ\right) = \text{undefined}$$

This indicates we're past the critical angle, so there is no transmitted ray.

Reflection

$$\theta_r = \theta_i = 55^\circ$$

And it will take all the energy of the incident ray, as well as be 'up', since bouncing off a lower n won't invert it.